# Lecture 29

In this lecture we continue our study of probability, looking at conditional probability and independent events.

# <u>Conditional Probability</u>

We often speak of "the probability of B given A".

Here, A and B are events or outcomes. If A has taken place, what is the probability that B has taken place?

For example, suppose we cast a die. We don't see the number, but someone else does and that person tells us that it was $> 3$. Then we could consider this question. Given that the score is $> 3$, what's the probability that it is an even number? This is an example of a <u>conditional</u> probability.

Here, $A$ is the outcome "$> 3$" and $B$ is the outcome "an even number".

Here's the general formula. It depends on $Pr(A)$ being nonzero.

Then:

$$Pr(B|A) = \frac{Pr(A \cap B)}{Pr(A)}$$

Let's apply this to the current problem.

Since $A$ is the outcome "$> 3$", we can write $A = \{4, 5, 6\}$.

Since $B$ is the outcome "an even number", we can write $B = \{2, 4, 6\}$.

Then $A \cap B = \{4, 6\}$.

So

$$Pr(B|A) = \frac{2/6}{3/6} = \frac{2}{3}.$$

The notion of conditional probability helps us to understand another concept — that of the <u>independence</u> of two events (in relation to each other).

Events $A$ and $B$ are <u>independent</u> if

$$\Pr(A \cap B) = \Pr(A) \cdot \Pr(B).$$

When $\Pr(A) \neq 0$, an equivalent way to describe independence is this:

$$\Pr(B \mid A) = \Pr(B)$$

This says that the probability of $B$ isn't affected by whether or not $A$ has taken place.

<u>E.g.</u> Suppose we toss a coin and cast a die.

Let $A$ be the outcome of $H$ on the coin toss. Let $B$ be the outcome of 3 on the die casting.

In set-theoretic terms we have the following:

The sample space is

$$S = \{(H,1), (H,2), \ldots, (H,6), \\ (T,1), (T,2), \ldots, (T,6)\}.$$

Then:

$$A = \{(H,1), (H,2), \ldots, (H,6)\}$$

$$B = \{(H,3), (T,3)\}$$

$$\therefore A \cap B = \{(H,3)\}$$

So:

$$Pr(A) = \frac{6}{12} = \frac{1}{2}$$

$$Pr(B) = \frac{2}{12} = \frac{1}{6}$$

$$Pr(A \cap B) = \frac{1}{12}$$

So
$$\Pr(A \cap B) = \Pr(A) \cdot \Pr(B)$$
and we can conclude that the two events are independent.

Actually, it seems intuitively obvious that the outcome of the coin toss won't affect the outcome of the die casting. Our calculations confirm this independence.

If we look back to the previous example, we can see that the two events were not independent. In the set $\{1, 2, 3, 4, 5, 6\}$ of die outcomes, there are just as many even numbers as odd numbers. But if we restrict the set to $\{4, 5, 6\}$ then even numbers predominate.

E.g. Suppose we throw two dice. Are the following two outcomes dependent or independent — a 4 on the first toss, and a 4 on the second toss?

Sol$^{\underline{n}}$ The sample space $S$ is the set of all ordered pairs in which both coordinates come from the set $\{1, \ldots, 6\}$.

If $A$ means "a 4 on the first toss", then $A$ is the 6-element set

$\{(4,1), (4,2), \ldots, (4,6\}$.

If $B$ means "a 4 on the second toss", then $B$ is the 6-element set

$\{(1,4), (2,4), \ldots, (6,4)\}$.

So $A \cap B = \{(4,4)\}$.

Now we do the probabilities:

$$\Pr(A) = \frac{6}{36} = \frac{1}{6}$$

$$\Pr(B) = \frac{6}{36} = \frac{1}{6}$$

$$\Pr(A \cap B) = \frac{1}{36}$$

So $\Pr(A \cap B) = \Pr(A) \cdot \Pr(B)$, and we conclude that the outcomes are independent.